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Abstract

We study how alternative market structures influence market supply and R&D investment decisions of firms operating in dynamic imperfectly competitive environments. Firms can reduce their future production cost through R&D investment today, which is the engine of endogenous industry growth. Our framework enables us to identify key strategic ingredients in firms' dynamic competitive behavior through analytical characterizations. These ingredients are a static market externality, stemming from the standard oligopolistic Cournot competition, a dynamic externality that arises due to knowledge spillovers, and a dynamic market externality that comes from the interaction of knowledge spillovers with future market oligopolistic competition that firms internalize while making decisions. We isolate the impact of each strategic ingredient by comparing four alternative market structures.

Key Words: R&D investment, Cournot competition, oligopolistic non-cooperative dynamic games

JEL classification: D43, D92, L13, O32

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1. Introduction

Investment in Research and Development (R&D) by firms is widely considered as one of the major engines of industry growth. A crucial type of R&D investment is cost innovation, i.e. investment that aims at reducing the production cost per unit of output. By undertaking a cost of investing in production knowledge today, a firm can reduce its future production cost and increase its future profit margin.

The possibilities for a firm to increase its profit margin through R&D investment are influenced by the existing market structure of the industry. For example, a monopoly may enjoy a high profit margin today, which may give the chance for high R&D investment, in order to achieve more profits in the future. If, however, monopolistic rents are already high enough, the monopoly may not have a strong incentive to undertake the cost of increasing this profit margin at a high rate, relative to other industrial structures.

On the other hand, Cournot competition in the final goods market may pressure duopolistic firms to increase their future profit margin by undertaking more R&D investment. Or, duopolistic firms, due to intense competition, may end up with little resources to finance enough R&D investment. Such conflicting incentives for R&D investment call for a theoretical treatment of the strategic elements that arise in different market structures.

Early literatures outlined by Kamien and Schwartz (1975) and Reingaum (1984) reveal that the question we pose in this paper is a very old one. Examples of studies focusing on the link between market structures and the intensity of lump-sum R&D spending are Loury (1979), Dasgupta and Stiglitz (1980), and Lee and Wilde (1980). These studies suggest that innovation effort of each firm declines with competition, yet a larger number of firms in an industry may lead to higher aggregate R&D spending. Recent empirical work testing the conclusions of such models about the link between market structure and innovation intensity suggests that competition and innovation have an inverted U relationship.[1]

In contrast with previous work that focuses on lump-sum R&D spending decisions in order to create or enter a market, we study R&D investment as a flow variable that can improve existing production cost smoothly. We suggest that firms operating in a fixed market structure and having the chance to cost-innovate at any point in time face different strategic dilemmas compared to these in cases where R&D is a lump-sum spending decision. In particular, inter-temporal considerations, like these stressed by McDonald and Siegel (1986) about monopolistic R&D investment, are linked up with intra-temporal Cournot-competition mechanics of oligopoly. Thus, our work relates to the dynamic-game approach of Mirman (1979) and Levhari and Mirman (1980).

We link R&D investment to industry growth directly in a deterministic framework.[2] A firm faces certain growth opportunities and its inter-temporal allocation of resources is affected by its ability to influence industry growth.

In order to study the link between market structure and growth-enhancing R&D investment, we identify and isolate, one by one, the strategic components of firm market supply and R&D investment in four alternative market structures. After studying a monopoly as a benchmark case, we compare it with alternative market structures that add a new strategic element each time.

The presence of another firm in the market gives rise to the usual Cournot-type *static market externality.* In order to isolate the impact of the static market externality on R&D investment strategies of firms and industry dynamics, we study a setup in which two symmetric firms accumulate knowledge that is protected by patents. With patents, knowl-

[1] See, for example, Aghion et. al. (2002). Moreover, another example of theoretical literature stressing the importance of the Schumpeterian idea of "creative destruction," for innovation is this of Aghion and Howitt (1992).

[2] Adding uncertainty is an extension that would not alter our main conclusions in this study.

edge is a perfectly excludable and rivalrous good. The comparison between the benchmark monopoly and this duopolistic setup is made in section 3. We find that key parameters of the model, like the demand elasticity determine the link between the static market externality and industry growth: a low demand elasticity induces a more intensive Cournot quantity competition that leaves little space for R&D investment, leading to lower growth.

The law of motion of knowledge of the benchmark monopoly changes if there are is another firm that undertakes R&D investment and there is a knowledge spillover, i.e. if accumulated knowlegde is non-excludable and non-rivalrous. When other firms also invest in knowledge, because the knowledge spillover directly affects the dynamic equation of knowledge accumulation, we say that the knowledge spillover is a *dynamic externality*. The impact of the dynamic externality on firm behavior is twofold, it will affect, (i) the future R&D return, and (ii) the future supply quantity. By examining a market structure of two monopolists with knowledge spillovers and by comparing this organization with the pure monopoly we isolate the influence of the dynamic externality on industry growth. We present this analysis in section 4. We find that the impact of the dynamic externality depends on parameter values that determine whether firms do or do not free-ride on each other for the benefits of innovation.

When two firms are in the same market and there are also knowledge spillovers, both the supply strategy and next period's knowledge stock are directly influenced by the presence of each other. Firms project and internalize the future interplay between the static market externality and the dynamic externality. This means that a more complex externality arises in this market structure, a *dynamic market externality*. In section 5 we study the role of this externality type in industry dynamics, finding that it enhances industry growth.

Our analysis is similar to this in Koulovatianos and Mirman (2003), in which the focus is the optimal utilization of industry-specific capital in different market structures. Even though both the economic question and setup in Koulovatianos and Mirman (2003) are different from these in this this study, the static market externality and the dynamic externality are, again, the main determinants of the link between market structure and industry growth.

A problem with dynamic games is that there are no widely-known general modeling conditions guaranteeing equilibrium existence or leading to certain properties of strategies.[3] This is the reason why we present a parametric model that enables us to have analytical characterizations. Our goal is to have a tractable model that emphasizes the economic behavior of firms in various market structures. Moreover, the parametric model of our study is a testable framework, despite its parameter restrictions. These parameter restrictions can be handled through data-mining approaches in empirical work.

2. The Benchmark Monopoly Model

Consider a firm operating in infinite horizon, $t = 0, 1, ...$, facing an inverse demand function

$$p_t = D(q_t) \ ,$$

for all $t = 0, 1, ...$ The cost function depends on the quantity of the produced final good and the stock of knowledge possessed by the firm with respect to production technology,

$$c_t = C(k_t)\, q_t \ ,$$

where k_t is the accumulated stock of knowledge at time t. The cost function has $C' < 0$, i.e. the higher the stock of knowledge, the lower the production cost per unit of final good.

[3] Some theoretical work on dynamic games includes Dutta and Sundaram (1992) and (1993). On the other hand, some work about computing dynamic games has been developed, like this of Vedenov and Miranda (2001) and Pakes and McGuire (2001), whereas Ericson and Pakes (1995) show the importance of Markov-perfect dynamics of dynamic games for empirical work.

In fact, several recent papers still deal with the issue of existence and uniqueness of Cournot-Nash equilibrium in *static* frameworks. See, for example, Gaudet and Salant (1991), Novshek (1984a), (1984b) and (1985).

Each period, the monopolistic firm can invest in knowledge. With x_t denoting R&D spending in period t, the firm accumulates knowledge according to the rule,

$$k_{t+1} = k_t + f(x_t) \ , \tag{1}$$

with $f'(x) > 0$ for all $x \geq 0$. The objective of the firm is to determine a final-good supply rule $q_t = Q(k_t)$ and an R&D investment rule $x_t = X(k_t)$ for $t = 0, 1, ...$, so that it maximizes its life-time profits,

$$\sum_{t=0}^{\infty} \delta^t \left[D(q_t) q_t - C(k_t) q_t - x_t \right] \ , \tag{2}$$

under the constraint (1) with $\delta \equiv \frac{1}{1+r}$, where $r > 0$ is a constant interest rate, and given an initial knowledge stock $k_0 > 0$.

The problem of the firm written in a Bellman-equation form is,

$$V_M(k) = \max_{q,x \geq 0} \left\{ D(q) q - C(k) q - x + \delta V_M(k + f(x)) \right\} \ . \tag{3}$$

Our goal is to obtain and characterize closed-form results for all market setups in this paper. For this reason, we use a specific parametric version of the model that enables us to have solutions of the form $Q(k) = \omega k$ and $X(k)$ such that $f(X(k)) = \gamma k$. In particular, the inverse demand function is given by,

$$D(q) = q^{-\frac{1}{\eta}} \ , \tag{4}$$

i.e. it is a constant-elasticity demand function with $\eta > 1$. The cost function is,

$$C(k) q = \nu k^{-\frac{1}{\eta}} q \ , \tag{5}$$

with $\nu > 0$, and the knowledge production function is

$$f(x) = x^{\frac{\eta}{\eta - 1}} \ , \tag{6}$$

reflecting the reasonable assumption that R&D investment exhibits increasing returns in knowledge production.

Although the presence of parameter η in all functions (4), (5), and (6) seems restrictive, these functional forms enable us to obtain linear decision rules for all cases of monopoly and duopoly throughout this paper. These linear rules are easy to characterize and in this way we can isolate, one by one, each and every strategic aspect of firm decisions. The decision rules will be of the form,

$$q = \omega k \ , \tag{7}$$

and

$$x^{\frac{\eta}{\eta-1}} = \gamma k \ , \tag{8}$$

with $\omega, \gamma > 0$. Substituting the decision rules (7) and (8) into the objective function of the monopolist and into the law of motion given by (1), we arrive at the following value function:

$$V_M(k) = \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \gamma^{1-\frac{1}{\eta}}}{1-\delta(1+\gamma)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} \ . \tag{9}$$

In order to identify the conditions that characterize ω and γ we must form the Bellman equation using (9). This is given by:

$$V_M(k) = \max_{q,x\geq 0}\left[q^{1-\frac{1}{\eta}} - \nu k^{-\frac{1}{\eta}} q - x + \delta \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \gamma^{1-\frac{1}{\eta}}}{1-\delta(1+\gamma)^{1-\frac{1}{\eta}}}\left(k + x^{\frac{\eta}{\eta-1}}\right)^{1-\frac{1}{\eta}}\right] \ .$$

The first-order conditions imply that,

$$q = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} k \ ,$$

so from (7) it is,

$$\omega_M = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} \ , \tag{10}$$

and

$$1 = \delta \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \gamma^{1-\frac{1}{\eta}}}{1-\delta(1+\gamma)^{1-\frac{1}{\eta}}}\left(k + x^{\frac{\eta}{\eta-1}}\right)^{-\frac{1}{\eta}} x^{\frac{1}{\eta-1}} \ . \tag{11}$$

Substituting (10) and (8) into (11), we obtain the condition that characterizes γ, namely,

$$g_M(\gamma) \equiv \left[(1+\gamma)^{\frac{1}{\eta}} - \delta\right]\gamma^{-\frac{1}{\eta}} = \delta\eta^{-\eta}\left(\frac{\eta-1}{\nu}\right)^{\eta-1} \equiv \kappa_M \ . \tag{12}$$

Proposition 1 *Under the parameter constraint,*

$$\eta^{-\frac{\eta^2}{\eta-1}}\left(\frac{\eta-1}{\nu}\right)^{\eta} + 1 > \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} \ , \tag{13}$$

there exists a unique strategy $x^{\frac{\eta}{\eta-1}} = \gamma_M k$*, such that* γ_M *satisfies equation (12) and such that the value function is bounded and the life-time profits are positive.*

Proof Substituting ω_M, as this is given by equation (10), into $V_M(k)$, it is,

$$V_M(k) = \frac{\eta^{-\eta}\left(\frac{\eta-1}{\nu}\right)^{\eta-1} - \gamma^{1-\frac{1}{\eta}}}{1-\delta(1+\gamma)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} \ .$$

The boundedness and positivity of $V_M(k)$ is secured if

$$\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1 \ , \tag{14}$$

and

$$\gamma < \eta^{-\frac{\eta^2}{\eta-1}}\left(\frac{\eta-1}{\nu}\right)^{\eta} \ . \tag{15}$$

The left-hand side of equation (12), $g_M(\gamma)$, is such that $g_M(0) = \infty$ and

$$g'_M(\gamma) = \frac{1}{\eta}\gamma^{-\frac{1}{\eta}-1}(1+\gamma)^{\frac{1}{\eta}-1}\left[\delta(1+\gamma)^{1-\frac{1}{\eta}} - 1\right] \ . \tag{16}$$

Equation (12) together with (14) imply that,

$$\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1 \Leftrightarrow g'_M(\gamma) < 0 \ . \tag{17}$$

Therefore, in order to ensure that there exists a $\gamma > 0$ satisfying both (14) and (15), it must be that

$$g_M\left(\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right) = \delta\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]^{1-\frac{1}{\eta}} < \delta\eta^{-\eta}\left(\frac{\eta-1}{\nu}\right)^{\eta-1} = \kappa_M \ .$$

The last inequality is equivalent to inequality (13) and it also implies (15) for $\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1$. The relationship (17) together with the intermediate value theorem imply that there exists a unique γ_M satisfying (12). This is depicted in Figure 1, which shows that $\gamma_M < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1$, which means also that (15) holds.□

The condition given by (13) has a direct economic interpretation. In equilibrium, the momentary profits of the firm, gross of its R&D investment cost, are equal to,

$$\left[\eta^{-\frac{\eta^2}{\eta-1}} \left(\frac{\eta-1}{\nu}\right)^{\eta} k\right]^{1-\frac{1}{\eta}} .$$

The term $\eta^{-\frac{\eta^2}{\eta-1}} \left(\frac{\eta-1}{\nu}\right)^{\eta}$ reflects the profitability of selling the final good in the market per unit of accumulated knowledge. Thus, meeting inequality (15) is equivalent to saying that the profitability of selling in the market should be greater than the R&D expense, linked to the growth rate of knowledge, γ. On the other hand, meeting inequality (14) means that the period-by-period cost of accumulating knowledge should not exceed the gross interest rate (associated with the discount factor, δ), which reflects the period-by-period opportunity cost of being in this industry. Inequality (13) guarantees that both (14) and (15) are met: in order to have positive bounded profits and R&D investment, the profitability of selling in the market should be high enough to cover the opportunity cost linked with the exogenous interest rate, r.

In contrast to a monopoly operating in a static environment, the dynamic monopoly takes into account the extra current cost of investing in R&D and its influence on the future cost of production per unit of final good, through the accumulation of knowledge. This is obvious from the firm's Euler equations. In particular, the first-order conditions implied by (3) are,

$$D(q) + D'(q)\, q = C(k) \ , \tag{18}$$

and

$$1 = \delta V_M' \left(\widehat{k}\right) f'(x) \ , \tag{19}$$

where $\widehat{k}$ is the stock of knowledge in the subsequent period. From equation (18) we can derive the firm's optimal quantity as a function of knowledge, $q = Q_M(k)$. According to equation (19), the monopolist equates the marginal cost of current R&D investment (which is equal to one) with the discounted change in next period's life-time profits due to a change in R&D investment in the current period. Applying the envelope theorem on the Bellman equation, it is,

$$V_M'(k) = -C'(k)\, q + \delta V_M' \left(\widehat{k}\right) \ . \tag{20}$$

The combination of the last two equations yields,

$$\delta f'(x) = \left[1 + \delta C' \left(\widehat{k}\right) Q_M \left(\widehat{k}\right) f'(x)\right] f'(\widehat{x}) \ , \tag{21}$$

where $\widehat{x}$ is the R&D expenditure in the subsequent period and the function $Q_M\left(\widehat{k}\right)$ gives the supply of the final good in the subsequent period which complies with (18). The Euler equation (21) implies that the firm counterbalances the current marginal product of R&D in terms of knowledge units (given by the term $f'(x)$), with the future marginal knowledge product as this interacts with the potential to reduce the final-good production cost in the future. The marginal reduction of next period's final-good production cost is captured by the term $C'\left(\widehat{k}\right) Q_M\left(\widehat{k}\right) f'(x)$, which has a negative sign. The economic interpretation of the term $-C'\left(\widehat{k}\right) Q_M\left(\widehat{k}\right) f'(x)$ is straightforward: it is *the marginal return of R&D investment between the current period and one period ahead.*

Remembering that $\delta = \frac{1}{1+r}$, the Euler equation (21) can be written as

$$f'(x) = \left[1 + r + C' \left(\widehat{k}\right) Q_M \left(\widehat{k}\right) f'(x)\right] f'(\widehat{x}) \ . \tag{22}$$

This last expression reminds a typical necessary condition in optimal-growth models dealing with the optimal intertemporal management of resources. The term $C'\left(\widehat{k}\right) Q_M \left(\widehat{k}\right) f'(x)$ having a negative sign, shows the potential for increasing monopolistic profits by taking actions not only in the market today, but also by taking actions that influence the future potential for profit making. Spending R&D money today has an opportunity cost: this money could be invested elsewhere, yielding r return per dollar. But reducing production costs tomorrow gives a profit margin that can compensate for the R&D spending. Thus, after incorporating this profit margin created by the reduction of future cost, the term $1+r+C'\left(\widehat{k}\right) Q_M \left(\widehat{k}\right) f'(x)$ is the gross effective interest rate paid for investing in knowledge, the opportunity cost of a dollar today minus the marginal return of a dollar invested in R&D. So, the interpretation of the monopolist's Euler equation is: *the optimal R&D management of the firm necessitates that its marginal knowledge output today, $f'(x)$, should equal its marginal knowledge output tomorrow, multiplied by the gross effective interest rate paid for investing in knowledge.*

2.1 The pure monopolist's Euler equation as a basis for comparison of alternative market structures

An interesting aspect of the Euler equation is the presence of the term $Q_M\left(\widehat{k}\right)$. This term, complying with equation (18), captures the fact that the market organization influences the R&D strategy. This is the gist of our analysis: to examine how the market structure influences R&D investment and, consequently, industry growth. The nature of the Euler equation, which is also present in all other market organizations that we examine in this paper, facilitates a clear understanding of the link between the market structure and industry growth.

The presence of another firm in the market gives rise to the usual Cournot-type *static* *market externality.* This externality is captured by the way the supply function, $Q_M(\cdot)$, changes. In order to isolate the impact of the static market externality on R&D investment strategies of firms and industry dynamics, we study a setup in which two symmetric firms accumulate knowledge that is protected by patents. With patents, knowledge is a perfectly excludable and rivalrous good. The comparison between the benchmark monopoly and this duopolistic setup is made in the section that follows.

The law of motion of knowledge of the benchmark monopoly, $\widehat{k} = k + f(x)$, changes if there are is another firm that undertakes R&D investment and there is a knowledge spillover, i.e. if accumulated knowlegde is non-excludable and non-rival. If, for example, there are two identical firms, A and B, both investing in R&D and there are knowledge spillovers, the law of motion of knowledge will be,

$$\widehat{k} = k + f(x_A) + f(x_B) \ , \tag{23}$$

where x_A and x_B are the levels of R&D spending by the two firms. When other firms also invest in knowledge, because the knowledge spillover directly affects the dynamic equation (23), we say that the knowledge spillover is a *dynamic externality.* The impact of the dynamic externality on firm behavior is twofold, it will affect, (i) the future R&D return, and (ii) the future supply quantity. These influences will be reflected in the necessary optimal conditions of each firm, and these conditions are very similar to equation (21).

If firms A and B are monopolists, equation (18) implies that the supply function, $Q_M(\cdot)$, will not change. So, by examining a market structure of two monopolists with knowledge spillovers, we learn about the influence of the dynamic externality, alone, on industry growth. We present this analysis in section 4.

When firms A and B are in the same market and there are also knowledge spillovers, both the supply function, $Q_M(\cdot)$, and next period's knowledge stock, the input of this new supply function, change. Thus, the necessary conditions of each firm capture the fact that changing the stock of knowledge tomorrow gives different potential profit margins when another firm is also in the market. This element in dynamic oligopoly implies that the presence of another firm in the market that also invests in non-excludable and non-rivalrous knowledge, will give rise to a *dynamic market externality.* Section 5 studies the role of this externality type in industry dynamics.

3. Duopoly with firms accumulating patent-protected knowledge

The goal of this section is to examine the impact of the static market externality on knowledge accumulation. We isolate the impact of Cournot competition on R&D investment strategies and industry dynamics. In order to achieve this goal, we examine a setup where two symmetric firms sell their final good in the same market, but each of the two firms has access to its own stock of knowledge that each firm protects through patents. For simplicity, we assume that there is no cost for obtaining a patent, there is only cost for obtaining knowledge, not for protecting it. While we have been using the subscript "M" to denote the pure monopoly value function and decision rules, in this section we use the subscript "D" in order to denote this market structure, the duopoly in which firms protect their knowledge stock by patents.

Consider two symmetric firms, A and B, selling the final good they produce in the same market. The two firms face the demand function

$$D\left(q_t\right) = q_t^{-\frac{1}{\eta}} \ ,$$

for $t = 0, 1, ...$, with $\eta > 1$. Both firms start from the same level of knowledge in period 0, that they protect by patents, i.e. $k_{A,0} = k_{B,0} > 0$. The two firms are symmetric also with respect to their cost functions The cost function is,

$$C\left(k_{A,t}\right)q_{A,t}=\nu k_{A,t}^{-\frac{1}{\eta}}q_{A,t}\ ,$$

for firm A, and

$$C\left(k_{B,t}\right)q_{B,t}=\nu k_{B,t}^{-\frac{1}{\eta}}q_{B,t}\ ,$$

for firm B, for $t=0,1,...,$ with $\nu>0$. The knowledge production function for both firms is given by equation (6), so the law of motion of capital for both firms is given by,

$$k_{A,t+1}=k_{A,t}+x_{A,t}^{\frac{\eta}{\eta-1}}\ , \tag{24}$$

and

$$k_{B,t+1}=k_{B,t}+x_{B,t}^{\frac{\eta}{\eta-1}}\ . \tag{25}$$

Due to the symmetry of the two firms, we can focus on the behavior of firm A, without loss of generality. The objective of firm A is to determine a final-good supply rule $q_{A,t}=Q_A\left(k_{A,t},k_{B,t}\right)$ and an R&D investment rule $x_{A,t}=X_A\left(k_{A,t},k_{B,t}\right)$ for $t=0,1,...,$ so that it maximizes its life-time profits,

$$\sum_{t=0}^{\infty}\delta^t\left[D\left(q_{A,t}+Q_B\left(k_{A,t},k_{B,t}\right)\right)q_{A,t}-C\left(k_{A,t}\right)q_{A,t}-x_{A,t}\right]\ , \tag{26}$$

under the constraint (24) and the strategies $Q_B\left(k_{A,t},k_{B,t}\right)$ and $X_B\left(k_{A,t},k_{B,t}\right)$ of firm B, given the law of motion (25), with $\delta\equiv\frac{1}{1+r}$, where $r>0$ is the constant interest rate, and given the initial knowledge stocks $k_{A,0}=k_{B,0}>0$.

The problem of firm A, written in a Bellman-equation form is,

$$\begin{aligned}V_{A,D}\left(k_A,k_B\right) &= \max_{q_A,x_A\geq 0}\Big\{D\left(q_A+Q_B\left(k_A,k_B\right)\right)q_A-C\left(k_A\right)q_A-x_A+\\ &\qquad +\delta V_{A,D}\Big(k_A+f\left(x_A\right)\ ,\ k_B+f\left(X_B\left(k_A,k_B\right)\right)\Big)\Big\}\,. \end{aligned} \tag{27}$$

Using the specific parametric version of the model given by equations (4), (5), and (6) enables us to have solutions of the form

$$Q_A(k_A, k_B) = \omega k_A \ , \tag{28}$$

$$Q_B(k_A, k_B) = \omega k_B \ , \tag{29}$$

$$[X_A(k_A, k_B)]^{\frac{\eta}{\eta-1}} = \gamma k_A \ , \tag{30}$$

and

$$[X_B(k_A, k_B)]^{\frac{\eta}{\eta-1}} = \gamma k_B \ . \tag{31}$$

Substituting the decision rules (28), (29), (30), and (31), into the objective function of firm A, as this is given by (26) and into the laws of motion given by (24) and (25), we arrive at the following value function,

$$V_{A,D}(k_A, k_B) = \frac{\omega^{1-\frac{1}{\eta}}(k_A + k_B)^{-\frac{1}{\eta}} k_A - \left[\nu\omega + \gamma^{1-\frac{1}{\eta}}\right] k_A^{1-\frac{1}{\eta}}}{1 - \delta(1+\gamma)^{1-\frac{1}{\eta}}} \ . \tag{32}$$

In order to identify the conditions that characterize ω and γ we must form the Bellman equation using (32). This is given by

$$V_{A,D}(k_A, k_B) = \max_{q_A, x_A \geq 0} \Bigg\{ [q_A + Q_B(k_A, k_B)]^{-\frac{1}{\eta}} q_A - \nu k_A^{-\frac{1}{\eta}} q_A - x_A +$$

$$+\delta \frac{\omega^{1-\frac{1}{\eta}} \left\{k_A + x_A^{\frac{\eta}{\eta-1}} + k_B + [X_B(k_A, k_B)]^{\frac{\eta}{\eta-1}}\right\}^{-\frac{1}{\eta}} \left(k_A + x_A^{\frac{\eta}{\eta-1}}\right) - \left[\nu\omega + \gamma^{1-\frac{1}{\eta}}\right]\left(k_A + x_A^{\frac{\eta}{\eta-1}}\right)^{1-\frac{1}{\eta}}}{1 - \delta(1+\gamma)^{1-\frac{1}{\eta}}} \Bigg\} \ .$$

The first-order conditions are,

$$(q_A + q_B)^{-\frac{1}{\eta}} \left(1 - \frac{1}{\eta} \frac{q_A}{q_A + q_B}\right) = \nu k_A^{-\frac{1}{\eta}} \ , \tag{33}$$

and

$$1=\frac{\delta\left\{\left(\widehat{k}_A+\widehat{k}_B\right)^{-\frac{1}{\eta}}\frac{\eta}{\eta-1}\left(1-\frac{1}{\eta}\frac{\widehat{k}_A}{\widehat{k}_A+\widehat{k}_B}\right)\omega-\left[\nu\omega+\gamma^{1-\frac{1}{\eta}}\right]\widehat{k}_A^{-\frac{1}{\eta}}\right\}x_A^{\frac{1}{\eta-1}}}{1-\delta\left(1+\gamma\right)^{1-\frac{1}{\eta}}}\ , \tag{34}$$

where $\widehat{k}_A$ and $\widehat{k}_B$ are the knowledge stocks of the two firms one period ahead. The Bellman equation and the first-order conditions of firm B are the same as these of firm A, with the roles of A and B switched. The symmetry of the fundamentals of the two firms, the symmetry of the firms' necessary optimal conditions, and the fact that also $k_{A,0}=k_{B,0}=k_0>0$, imply that the knowledge stock of the two firms is the same at all times. This observation cross validates the symmetric solution of the form $Q_A\left(k_A,k_B\right)=\omega_D k_A$ and $Q_B\left(k_A,k_B\right)=\omega_D k_B$, in which, using equation (33),

$$\omega_D=\frac{1}{2}\left(\frac{1-\frac{1}{2\eta}}{\nu}\right)^{\eta}\ . \tag{35}$$

Moreover, symmetry also implies the validity of the strategies $\left[X_A\left(k_A,k_B\right)\right]^{\frac{\eta}{\eta-1}}=\gamma k_A$ and $\left[X_B\left(k_A,k_B\right)\right]^{\frac{\eta}{\eta-1}}=\gamma k_B$. Substituting (35) and (30) into (34), we obtain the condition that characterizes γ, namely,

$$g_D\left(\gamma\right)\equiv\left[\left(1+\gamma\right)^{\frac{1}{\eta}}-\delta\right]\gamma^{-\frac{1}{\eta}}=\delta\frac{\nu^{1-\eta}}{\eta-1}\left[\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta}\right]\equiv\kappa_D\ . \tag{36}$$

Proposition 2 *Under the parameter constraint,*

$$\frac{\nu^{1-\eta}}{2\eta-1}\left[\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta}\right]>\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}}-1\right]^{1-\frac{1}{\eta}}\ , \tag{37}$$

there exists a unique couple of strategies $x_A^{\frac{\eta}{\eta-1}}=\gamma_D k_A$ and $x_B^{\frac{\eta}{\eta-1}}=\gamma_D k_B$, such that γ_D satisfies equation (36) and such that the value function of each firm is bounded and the life-time profits of each firm are positive.

Proof Substituting ω_D as given by equation (35) and the symmetry condition $k_A=k_B=k$ into the momentary-profit function of firm A, it is,

$$\pi_{A,D}\left(k,k\right)=\left\{\frac{\nu^{1-\eta}}{2\eta-1}\left[\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta}\right]-\gamma^{1-\frac{1}{\eta}}\right\}k^{1-\frac{1}{\eta}}\ ,$$

so, positivity of profits requires that,

$$\gamma^{1-\frac{1}{\eta}} < \frac{\nu^{1-\eta}}{2\eta-1}\left[\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta}\right] . \tag{38}$$

At the same time, the boundedness of $V_{A,D}(k_A, k_B)$ is secured if

$$\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1 . \tag{39}$$

The left-hand side of equation (36), $g_D(\gamma)$, is such that $g_D(0) = \infty$ and

$$g_D'(\gamma) = \frac{1}{\eta}\gamma^{-\frac{1}{\eta}-1}(1+\gamma)^{\frac{1}{\eta}-1}\left[\delta(1+\gamma)^{1-\frac{1}{\eta}} - 1\right] .$$

We can see that (39) together with (36) imply that,

$$\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1 \Leftrightarrow g_D'(\gamma) < 0 . \tag{40}$$

Therefore, in order to ensure that there exists a $\gamma > 0$ satisfying both (39) and (38), it must be that

$$g_D\left(\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right) = \delta\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]^{1-\frac{1}{\eta}} < \delta\frac{\nu^{1-\eta}}{\eta-1}\left[\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta}\right] = \kappa_D .$$

The last inequality is equivalent to inequality (37) and it also implies (38) for $\gamma < \left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1$. The relationship (40) together with $g_D(0) = \infty$, through the intermediate value theorem imply that there exists a unique γ_D satisfying (36).□

The economic interpretation of the parameter constraint (37) is, again, that the profitability of selling in the market should be high enough to cover the opportunity cost associated with the exogenous interest rate, r, and also to leave space for undertaking R&D investment.

Each firm's optimality conditions reveal the differences from the pure monopoly case. The first-order conditions implied by (27), for firm A, are,

$$D(q_A + q_B) + D'(q_A + q_B)\, q_A = C(k_A) \ , \tag{41}$$

and

$$1 = \delta \frac{\partial V_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)}{\partial \widehat{k}_A} f'\left(x_A\right) \ , \tag{42}$$

where $\widehat{k}_A$ and $\widehat{k}_B$ are the stocks of knowledge in the subsequent period. Applying the envelope theorem on the Bellman equation (27), it is,

$$\frac{\partial V_{A,D}\left(k_A, k_B\right)}{\partial k_A} = -C'\left(k_A\right) q_A + D'\left(q_A + Q_B\left(k_A, k_B\right)\right) q_A \frac{\partial Q_B\left(k_A, k_B\right)}{\partial k_A} + \\ +\delta \left[\frac{\partial V_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)}{\partial \widehat{k}_A} + \frac{\partial V_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)}{\partial \widehat{k}_B} f'\left(x_B\right) \frac{\partial X_B\left(k_A, k_B\right)}{\partial k_A} \right] \ . \tag{43}$$

The terms $D'\left(q_A + Q_B\left(k_A, k_B\right)\right) q_A \frac{\partial Q_B(k_A, k_B)}{\partial k_A}$ and $\delta \frac{\partial V_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)}{\partial \widehat{k}_B} f'\left(x_B\right) \frac{\partial X_B(k_A, k_B)}{\partial k_A}$ in condition (43) are the new strategic elements compared to the coreesponding condition of the pure monopoly. But in the context of this model with symmetric strategies, as these are given by (28) through (31), $\frac{\partial Q_B(k_A, k_B)}{\partial k_A} = \frac{\partial X_B(k_A, k_B)}{\partial k_A} = 0$, so these terms vanish. Therefore, it is:

$$\frac{\partial V_{A,D}\left(k_A, k_B\right)}{\partial k_A} = -C'\left(k_A\right) q_A + \delta \frac{\partial V_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)}{\partial \widehat{k}_A} \ .$$

The combination of the last two equations, taking into account that $\delta = \frac{1}{1+r}$, yields,

$$f'\left(x_A\right) = \left[1 + r + C'\left(\widehat{k}_A\right) Q_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right) f'\left(x_A\right)\right] f'\left(\widehat{x}_A\right) \ . \tag{44}$$

where $\widehat{x}_A$ is the R&D expenditure in the subsequent period and the function $Q_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)$ gives the supply of the final good in the subsequent period which complies with (41) and $D\left(q_A + q_B\right) + D'\left(q_A + q_B\right) q_B = C\left(k_B\right)$, in the symmetric case of $k_A = k_B$. The term $1 + r + C'\left(\widehat{k}_A\right) Q_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right) f'\left(x_A\right)$ is the gross effective interest rate paid for investing in knowledge, the opportunity cost of a dollar today minus the marginal return of a dollar invested in R&D. So, the interpretation of equation (44) again is that the marginal knowledge output today, $f'\left(x_A\right)$, should equal the marginal knowledge output tomorrow, multiplied by the gross effective interest rate paid for investing in knowledge.

The most interesting aspect of equation (44) is that, compared to the pure monopolist, only the static market externality is added in the market structure of this section. This is reflected by the fact that $Q_{A,D}\left(\widehat{k}_A, \widehat{k}_B\right)$ differs from $Q_M\left(\widehat{k}\right)$, exactly in the way that they would differ in a static context. Yet, this seemingly simple departure from the pure monopoly benchmark changes the dynamics dramatically: with a different period-by-period profit margin due to oligopolistic competition, there are different opportunities for R&D investment.

3.1 The effect of the static market externality on market supply and industry growth

The comparison of the pure monopoly and the duopoly with patent protection of knowledge reveals the impact of the static market externality on industry growth. The nature of the linear quantity supply rules, that applies to both setups, enables a direct comparison.

If the level of knowledge of a monopolist equals the knowledge level of each of the two duopolists, then

$$Q_M(k) = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} k \ ,$$

and

$$Q_{A,D}(k,k) + Q_{B,D}(k,k) = \left(\frac{1-\frac{1}{2\eta}}{\nu}\right)^{\eta} k \ ,$$

imply that in the duopoly aggregate market supply is higher compared to the monopoly supply. But the growth rate of knowledge in each industrial setup determines the level of market supply in the long run.

The growth rate of the pure monopoly is given by the level of γ_M that solves equation (12), whereas the industry growth rate of the duopoly equals the level of γ_D determined by (36).

It is clear that

$$g_M(\gamma) = g_D(\gamma)$$

for all $\gamma \geq 0$. Moreover,

$$\kappa_D > \kappa_M \quad \text{if and only if} \quad \eta < 2.73 \ ,$$

since $\frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta} > \left(1-\frac{1}{\eta}\right)^{\eta}$ if and only if $\eta < 2.73$. Figures 2.a and 2.b depict the equilibrium R&D decision rules for the two setups, for the cases where $\eta < 2.73$ and $\eta > 2.73$. In all cases, parameters η, δ and ν must be such that the parameter constraints (13) and (37) are met. Figures 2.a and 2.b make clear that

$$\gamma_D < \gamma_M \quad \text{if and only if} \quad \eta < 2.73 \ .$$

As it is the case in a static environment with isoelastic demand, both duopolists always supply more *on aggregate*, compared to the monopolist in each period. But when $\eta < 2.73$, $\omega_D = \frac{1}{2}\left(1-\frac{1}{2\eta}\right)^{\eta} > \left(1-\frac{1}{\eta}\right)^{\eta} = \omega_M$, so, each duopolist *alone* supplies more compared to the monopolist, because the Cournot war is very intensive with low elasticity of demand. In the dynamic context of our model, these static mechanics have a direct influence on R&D investment and industry growth.

The low demand elasticity implies low marginal revenue for both the monopolist and the duopolists. The lower the demand elasticity, the less a firm can gain from price increases by contracting its supply. While playing their Cournot game with low elasticity, the duopolists can expand their market supply with lower revenue loss due to price decreases. But below the critical value 2.73 of η, the duopolists are carried away from their quantity competition so that their aggregate market supply lowers the price and each firm's profits to a level they do not have enough resources to finance as much R&D as the monopolist.

In the case of high demand elasticity, $\eta > 2.73$, for a given knowledge level, each duopolist alone supplies less of the final good compared to the monopolist, but the two duopolists supply more on aggregate. The necessary conditions depicted in figure 2.b imply that each monopolist alone will spend more resources on R&D investment compared to the monopolist. So, for $\eta > 2.73$, the duopolistic industry with patent-protected knowledge grows faster.

Our model points out that the elasticity of demand plays a critical role in determining the influence of the market externality on industry dynamics.[4] A low demand elasticity induces a more intensive intra-period quantity war for the duopoly that leaves little space for R&D investment, compared to the investment ability of a pure monopoly. Thus, industry growth in the duopoly setup is lower compared to this of a monopoly for low levels of demand elasticity. The opposite holds if the demand elasticity is high.

4. Two monopolies accumulating non-excludable and non-rivalrous knowledge

In this section we examine the impact of the dynamic externality on knowledge accumulation. Each firm is a monopolist in its own, separate market. Each firm spends on R&D, but it cannot exclude the other firm from using this knowledge, i.e. knowledge is non-excludable. Moreover, the other firm benefits as much as the inventor, i.e. knowledge is non-rivalrous. Thus, knowledge investment by one firm has a spillover effect on all others and this affects the future knowledge stock and production cost of all firms in the same way. While we have been using the subscript "M" to denote the pure monopoly value function and decision rules, in this section we use the subscript "m" in order to denote this special type of monopoly,

[4] The fact that we link the elasticity of demand with cost- and knowledge-production parameters is not important for drawing this qualitative conclusion, that the determinants of demand is the key to the role of the static market externality for industry growth. Looking at a static duopolistic model where the cost function is $C(k) = \nu k^{-\alpha}$, with $\alpha > 0$ and $\alpha \neq \frac{1}{\eta}$, the qualitative conclusions are the same for high and low values of η, simply the threshold level of η for each duopolist to supply more than the monopolist now depends on parameter α as well.

the monopoly with knowledge spillovers.

Consider two symmetric firms, A and B, selling the final good they produce in their own market. The two markets are identical, and each firm faces the demand function

$$D\left(q_t\right) = q_t^{-\frac{1}{\eta}} \ ,$$

for $t = 0, 1, ...,$ with $\eta > 1$, and both firms have the same cost function per unit of production,

$$C\left(k_t\right) = \nu k_t^{-\frac{1}{\eta}} \ ,$$

where k_t is the total stock of accumulated knowledge at time t. The knowledge production function for both firms is given by equation (6), so the law of motion of knowledge is given by,

$$k_{t+1} = k_t + x_{A,t}^{\frac{\eta}{\eta-1}} + x_{B,t}^{\frac{\eta}{\eta-1}} \ . \tag{45}$$

Due to the symmetry of the two firms, we can focus on the behavior of firm A, without loss of generality. The objective of firm A is to determine a final-good supply rule $q_{A,t} = Q_{A,m}\left(k_t\right)$ and an R&D investment rule $x_{A,t} = X_{A,m}\left(k_t\right)$ for $t = 0, 1, ...,$ so that it maximizes its life-time profits,

$$\sum_{t=0}^{\infty} \delta^t \left[D\left(q_{A,t}\right) q_{A,t} - C\left(k_t\right) q_{A,t} - x_{A,t}\right] \ , \tag{46}$$

under constraint (45) and the strategies $Q_{B,m}\left(k_t\right)$ and $X_{B,m}\left(k_t\right)$ of firm B, given the initial knowledge stock $k_0 > 0$. The interest rate is the same in both firms' markets, so the discount factor δ is the same for both firms.

The problem of firm A, written in a Bellman-equation form is,

$$V_{A,m}\left(k\right) = \max_{q_A, x_A \geq 0} \left\{D\left(q_A\right) q_A - C\left(k\right) q_A - x_A + \delta V_{A,m}\left(k + f\left(x_A\right) + f\left(\left[X_B\left(k\right)\right]\right)\right)\right\} \ . \tag{47}$$

Using the specific parametric version of the model given by equations (4), (5), and (6) enables us to have solutions of the form

$$Q_{A,m}\left(k\right) = Q_{B,m}\left(k\right) = \omega k \ , \tag{48}$$

and

$$[X_A(k)]^{\frac{\eta}{\eta-1}} = [X_B(k)]^{\frac{\eta}{\eta-1}} = \chi k \ . \tag{49}$$

Substituting the decision rules (48) and (49), into the objective function of firm A, given by (46), and using the law of motion given by (45), we arrive at the following value function,

$$V_{A,m}(k) = \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} \ . \tag{50}$$

In order to identify the conditions that characterize ω and χ we must form the Bellman equation using (50). This is given by

$$V_{A,m}(k) = \max_{q_A, x_A \geq 0} \left\{ q_A^{1-\frac{1}{\eta}} - \nu k^{-\frac{1}{\eta}} q_A - x_A + \delta \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} \left\{ k + x_A^{\frac{\eta}{\eta-1}} + [X_B(k)]^{\frac{\eta}{\eta-1}} \right\}^{1-\frac{1}{\eta}} \right\} \ .$$

The first-order conditions imply that,

$$q_A = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} k \ ,$$

so from (48) it is,

$$\omega_m = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} \ , \tag{51}$$

and

$$1 = \delta \frac{\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} \left\{ k + x_A^{\frac{\eta}{\eta-1}} + [X_B(k)]^{\frac{\eta}{\eta-1}} \right\}^{-\frac{1}{\eta}} x_A^{\frac{1}{\eta-1}} \ . \tag{52}$$

Substituting (51) and (49) into (52), we obtain the condition that characterizes χ, namely,

$$g_m(\chi) \equiv \left[(1+2\chi)^{\frac{1}{\eta}} - \delta(1+\chi)\right] \chi^{-\frac{1}{\eta}} = \delta\eta^{-\eta} \left(\frac{\eta-1}{\nu}\right)^{\eta-1} \equiv \kappa_m \ . \tag{53}$$

We stress that $\kappa_m = \kappa_M$.

Proposition 3 *Under the parameter constraint,*

$$\eta^{-\eta} \left(\frac{\eta-1}{\nu}\right)^{\eta-1} > \left\{ \frac{1}{2} \left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1 \right] \right\}^{1-\frac{1}{\eta}} \ , \tag{54}$$

there exists a unique strategy $x_A^{\frac{\eta}{\eta-1}} = \chi_m k$*, such that* χ_m *satisfies equation (53) and such that the value function is bounded and the life-time profits are positive.*

Proof Substituting ω_m from equation (51) into $V_m(k)$, it is,

$$V_m(k) = \frac{\eta^{-\eta}\left(\frac{\eta-1}{\nu}\right)^{\eta-1} - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} \ ,$$

and the boundedness and positivity of $V_m(k)$ is secured if

$$\chi < \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right] \ , \tag{55}$$

and

$$\chi < \eta^{-\frac{\eta^2}{\eta-1}}\left(\frac{\eta-1}{\nu}\right)^{\eta} \ . \tag{56}$$

The left-hand side of equation (53), $g_m(\chi)$, is such that $g_m(0) = \infty$ and

$$g_m'(\chi) < 0 \Leftrightarrow 1 - [1-(\eta-1)\chi]\,\delta(1+2\chi)^{1-\frac{1}{\eta}} > 0 \ . \tag{57}$$

But (55) implies that $1 - [1-(\eta-1)\chi]\,\delta(1+2\chi)^{1-\frac{1}{\eta}} > 0$, so from (57) it is,

$$\chi < \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right] \Leftrightarrow g_m'(\gamma) < 0 \ . \tag{58}$$

Moreover,

$$g_m\left(\frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]\right) = -\frac{\delta}{2}\left\{\frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]\right\}^{-\frac{1}{\eta}} < 0 \ .$$

Therefore, there exists a $\chi_m > 0$ satisfying (55), as it is shown by Figure 3. So, condition (54) secures that profits are positive even if $\chi_m = \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]$, so, since (55) holds, condition (56) holds as well.□

The parameter constraint (54) implies that the profitability of selling in the market should be high enough to cover the opportunity cost associated with the exogenous interest rate, r, and also to allow the undertaking of R&D investment.

The first-order conditions implied by (47), for firm A, are,

$$D(q_A) + D'(q_A)\,q_A = C(k) \ , \tag{59}$$

and

$$1 = \delta V'_{A,m}\left(\widehat{k}\right) f'\left(x_A\right) \ , \tag{60}$$

where $\widehat{k}$ is the stock of knowledge in the subsequent period. Applying the envelope theorem on the Bellman equation (47), it is,

$$V'_{A,m}\left(k\right) = -C'\left(k\right) q_A + \delta V'_{A,m}\left(\widehat{k}\right)\left[1 + f'\left(X_B\left(k\right)\right) X'_B\left(k\right)\right] \ . \tag{61}$$

The marginal life-time profit gains from an increase in the stock of knowledge are higher compared to the pure monopoly case (the equivalent equation for the pure monopoly is (20)). The marginal gains in the current period equal the sum of, (i) the decrease in the current production cost, captured by the term $-C'\left(k\right) q_A$, and (ii) the discounted marginal increase in next-period's life-time profits, but, in this setup, these are enhanced by the change in R&D investment of the other firm, due to the knowledge spillover. The term $f'\left(X_B\left(k\right)\right) X'_B\left(k\right)$ captures the marginal increase in next period's knowledge stock, $\widehat{k}$, due to investment by the other firm.

The combination of the last two equations, taking into account that $\delta = \frac{1}{1+r}$, yields,

$$f'\left(x_A\right) = \frac{1 + r + C'\left(\widehat{k}_A\right) Q_{A,m}\left(\widehat{k}\right) f'\left(x_A\right)}{1 + f'\left(X_B\left(\widehat{k}\right)\right) X'_B\left(\widehat{k}\right)} f'\left(\widehat{x}_A\right) \ . \tag{62}$$

where $\widehat{x}_A$ is the R&D expenditure in the subsequent period and the function $Q_{A,m}\left(\widehat{k}\right)$ gives the supply of the final good in the subsequent period which complies with (59). The term $\frac{1+r+C'\left(\widehat{k}_A\right)Q_{A,m}\left(\widehat{k}\right)f'\left(x_A\right)}{1+f'\left(X_B\left(\widehat{k}\right)\right)X'_B\left(\widehat{k}\right)}$ is the gross effective interest rate paid for investing in knowledge, the opportunity cost of a dollar today minus the marginal return of a dollar invested in R&D, discounted even more by the other firm's increase in R&D investment in the subsequant period. Noticing that $Q_{A,m}\left(\widehat{k}\right) = Q_M\left(\widehat{k}\right)$, with the latter being the supply strategy of the pure monopolist, we can see that the necessary optimality conditions (62) and (21) differ with respect to the discount term $\frac{1}{1+f'\left(X_B\left(\widehat{k}\right)\right)X_B'\left(\widehat{k}\right)}$. If the laws of motion for capital were the same for both setups, then the presence of the discount term $\frac{1}{1+f'\left(X_B\left(\widehat{k}\right)\right)X_B'\left(\widehat{k}\right)}$ would lead to more R&D investment in the setup of two monopolies with a knowledge spillover. However, the fact that the law of motion in the pure monopoly is $\widehat{k} = k + f(x)$, whereas the law of motion in the setup of this section is $\widehat{k} = k + f(x_A) + f(x_B)$, creates an incentive to invest less in R&D when the knowledge spillover is present. Thus, the influence of the dynamic externality on R&D investment and growth in the industry becomes ambiguous. Even in the context of our simple model, the effect of the dynamic externality depends on the values of its parameters.

4.1 The impact of the dynamic externality on market supply and industry growth

Conditions (10) and (51) imply that, for the same stock of knowledge in both monopolistic setups, the market supply of each monopolist is exactly the same. With respect to industry growth, the equilibrium R&D strategies $[X_A(k)]^{\frac{\eta}{\eta-1}} = [X_B(k)]^{\frac{\eta}{\eta-1}} = \chi_m k$ together with (45) imply that the growth rate of knowledge and of each monopolist's market supply is given by,

$$\gamma_m \equiv 2\chi_m \ , \tag{63}$$

where χ_m satisfies condition (53). On the other hand, the growth rate of knowledge and market supply of the pure monopoly is given by the γ_M that satisfies condition (12). In order to compare the two growth rates, we express condition (53) in terms of γ_m, using (63), namely,

$$2^{\frac{1}{\eta}} \left[(1+\gamma_m)^{\frac{1}{\eta}} - \delta \left(1 + \frac{1}{2}\gamma_m \right) \right] \gamma_m^{-\frac{1}{\eta}} = \kappa \ , \tag{64}$$

where $\kappa \equiv \kappa_M = \kappa_m = \delta\eta^{-\eta}\left(\frac{\eta-1}{\nu}\right)^{\eta-1}$. From (12), the pure monopoly's condition is,

$$\left[(1+\gamma_M)^{\frac{1}{\eta}} - \delta\right]\gamma_M^{-\frac{1}{\eta}} = \kappa \ . \tag{65}$$

Re-arranging the terms of (64), so that the left-hand sides of (64) and (65) are exactly the same, condition (64) becomes,

$$\left[(1+\gamma_m)^{\frac{1}{\eta}} - \delta\right]\gamma_m^{-\frac{1}{\eta}} = 2^{-\frac{1}{\eta}}\kappa + \frac{\delta}{2}\gamma_m \ . \tag{66}$$

It is easy to see by comparing (65) with (66), that if paramaters δ, η and ν are such that

$$\gamma_m^{1-\frac{1}{\eta}} = 2\left(1 - \frac{1}{2^{\frac{1}{\eta}}}\right)\frac{\kappa}{\delta} \ ,$$

then

$$\gamma_m = \gamma_M \ .$$

If δ, η and ν are such that

$$\gamma_m^{1-\frac{1}{\eta}} \lesseqgtr 2\left(1 - \frac{1}{2^{\frac{1}{\eta}}}\right)\frac{\kappa}{\delta} \ ,$$

then

$$\gamma_m \gtreqless \gamma_M \ .$$

In brief, when there is a knowledge spillover, compared to a pure monopoly, firms may or may not free-ride on each other with respect to R&D spending and industry growth may decrease or increase, depending on the parameter values. The impact of the dynamic externality on industry growth is ambiguous.

5. Duopoly accumulating non-excludable and non-rivalrous knowledge

The industrial setup of this section reveals the impact of the dynamic market externality on knowledge accumulation. There are two identical firms selling in the same market. Again, knowledge is non-excludable and non-rival. In this section we use the subscript "d" in order to denote this special type of duopoly, the duopoly with knowledge spillovers.

Consider two symmetric firms, A and B, selling their final good in a common market. The demand function is,

$$D(q_t) = q_t^{-\frac{1}{\eta}} ,$$

for $t = 0, 1, ...,$ with $\eta > 1$, and both firms have the same cost function per unit of production,

$$C(k_t) = \nu k_t^{-\frac{1}{\eta}} ,$$

where k_t is the total stock of accumulated knowledge at time t. The knowledge production function for both firms is given by equation (6), so the law of motion of knowledge is, again, given by,

$$k_{t+1} = k_t + x_{A,t}^{\frac{\eta}{\eta-1}} + x_{B,t}^{\frac{\eta}{\eta-1}} . \tag{67}$$

Due to the symmetry of the two firms, we can focus on the behavior of firm A, without loss of generality. The objective of firm A is to determine a final-good supply rule $q_{A,t} = Q_{A,d}(k_t)$ and an R&D investment rule $x_{A,t} = X_{A,d}(k_t)$ for $t = 0, 1, ...,$ so that it maximizes its life-time profits,

$$\sum_{t=0}^{\infty} \delta^t \left[D\left(q_{A,t} + Q_B(k_t)\right) q_{A,t} - C(k_t) q_{A,t} - x_{A,t} \right] , \tag{68}$$

under the constraint (67) and the strategies $Q_B(k_t)$ and $X_B(k_t)$ of firm B, given the initial knowledge stock $k_0 > 0$.

The problem of firm A, written in a Bellman-equation form is,

$$V_{A,d}(k) = \max_{q_A, x_A \geq 0} \left\{ D\left(q_A + Q_B(k)\right) q_A - C(k) q_A - x_A + \delta V_{A,d}\left(k + f(x_A) + f([X_B(k)])\right) \right\} . \tag{69}$$

Using the specific parametric version of the model given by equations (4), (5), and (6) enables us to have solutions of the form

$$Q_{A,d}(k) = Q_{B,d}(k) = \omega k \ , \tag{70}$$

and

$$[X_{A,d}(k)]^{\frac{\eta}{\eta-1}} = [X_{B,d}(k)]^{\frac{\eta}{\eta-1}} = \chi k \ . \tag{71}$$

Substituting the decision rules (70) and (71), into the objective function of firm A, as this is given by (68) and into the law of motion given by (67), we arrive at the following value function,

$$V_{A,d}(k) = \frac{2^{-\frac{1}{\eta}}\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} \ . \tag{72}$$

In order to identify the conditions that characterize ω and χ we must form the Bellman equation using (72), namely,

$$V_{A,d}(k) = \max_{q_A, x_A \geq 0} \left\{ [q_A + Q_B(k)]^{-\frac{1}{\eta}} q_A - \nu k^{-\frac{1}{\eta}} q_A - x_A + \right.$$

$$\left. +\delta \frac{2^{-\frac{1}{\eta}}\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} \left\{ k + x_A^{\frac{\eta}{\eta-1}} + [X_B(k)]^{\frac{\eta}{\eta-1}} \right\}^{1-\frac{1}{\eta}} \right\} \ .$$

The first-order conditions imply that,

$$q_A = \frac{1}{2}\left(\frac{1-\frac{1}{2\eta}}{\nu}\right)^{\eta} k \ ,$$

so from (70) it is,

$$\omega_d = \frac{1}{2}\left(\frac{1-\frac{1}{2\eta}}{\nu}\right)^{\eta} \ , \tag{73}$$

and

$$1 = \delta \frac{2^{-\frac{1}{\eta}}\omega^{1-\frac{1}{\eta}} - \nu\omega - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} \left\{ k + x_A^{\frac{\eta}{\eta-1}} + [X_B(k)]^{\frac{\eta}{\eta-1}} \right\}^{-\frac{1}{\eta}} x_A^{\frac{1}{\eta-1}} \ . \tag{74}$$

Substituting (73) and (71) into (74), we obtain the condition that characterizes χ, namely,

$$g_d(\chi) \equiv \left[(1+2\chi)^{\frac{1}{\eta}} - \delta(1+\chi)\right]\chi^{-\frac{1}{\eta}} = \delta\frac{\eta^{-\eta}}{2^{\eta+1}}\left(\frac{2\eta-1}{\nu}\right)^{\eta-1} \equiv \kappa_d \ . \tag{75}$$

We stress that $g_d = g_m$.

Proposition 4 *Under the parameter constraint,*

$$\frac{\eta^{-\eta}}{2^{\eta+1}}\left(\frac{2\eta-1}{\nu}\right)^{\eta-1} > \left\{\frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]\right\}^{1-\frac{1}{\eta}} , \tag{76}$$

there exists a unique strategy $x_A^{\frac{\eta}{\eta-1}} = \chi_d k$*, such that* χ_d *satisfies equation (75) and such that the value function is bounded and the life-time profits are positive.*

Proof Substituting ω_d as given by equation (73) into $V_{A,d}(k)$, it is,

$$V_d(k) = \frac{\frac{\eta^{-\eta}}{2^{\eta+1}}\left(\frac{2\eta-1}{\nu}\right)^{\eta-1} - \chi^{1-\frac{1}{\eta}}}{1-\delta(1+2\chi)^{1-\frac{1}{\eta}}} k^{1-\frac{1}{\eta}} ,$$

and the boundedness and positivity of $V_{A,d}(k)$ is secured if

$$\chi < \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right] , \tag{77}$$

and

$$\chi < \left(\frac{\eta^{-\eta}}{2^{\eta+1}}\right)^{\frac{\eta}{\eta-1}}\left(\frac{2\eta-1}{\nu}\right)^{\eta} . \tag{78}$$

Since $g_d = g_m$, the argument showing that

$$\chi < \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right] \Leftrightarrow g_d'(\gamma) < 0 \ . \tag{79}$$

is the same as in the proof of proposistion 4. The fact that,

$$g_d\left(\frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]\right) = -\frac{\delta}{2}\left\{\frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]\right\}^{-\frac{1}{\eta}} < 0 ,$$

shows that there exists a $\chi_d > 0$ satisfying (77), as it is shown by Figure 4. So, condition (76) secures that profits are positive even if $\chi_d = \frac{1}{2}\left[\left(\frac{1}{\delta}\right)^{\frac{\eta}{\eta-1}} - 1\right]$. Since (77) holds, condition (78) holds as well.□

The first-order conditions implied by (69) for firm A are,

$$D\left(q_A + Q_B\left(k\right)\right) + D'\left(q_A + Q_B\left(k\right)\right) q_A = C\left(k\right) \ , \tag{80}$$

and

$$1 = \delta V'_{A,d}\left(\widehat{k}\right) f'\left(x_A\right) \ , \tag{81}$$

where $\widehat{k}$ is the stock of knowledge in the subsequent period. Applying the envelope theorem on the Bellman equation (69), it is,

$$V'_{A,d}\left(k\right) = -C'\left(k\right) q_A + \delta V'_{A,d}\left(\widehat{k}\right)\left[1 + f'\left(X_B\left(k\right)\right) X'_B\left(k\right)\right] \ . \tag{82}$$

The combination of the last two equations, taking into account that $\delta = \frac{1}{1+r}$, yields,

$$f'\left(x_A\right) = \frac{1 + r + C'\left(\widehat{k}_A\right) Q_{A,d}\left(\widehat{k}\right) f'\left(x_A\right)}{1 + f'\left(X_B\left(\widehat{k}\right)\right) X'_B\left(\widehat{k}\right)} f'\left(\widehat{x}_A\right) \ . \tag{83}$$

where $\widehat{x}_A$ is the R&D expenditure in the subsequent period. The function $Q_{A,d}\left(\widehat{k}\right)$ gives the supply of the final good in the subsequent period which complies with (80).

The necessary condition of firm A, given by (83) has similarities with both the necessary condition of the two monopolies with knowledge spillovers, given by (62), and with the necessary condition of the duopoly with patent protection, given by (44). The similarity with the setup of two monopolies with knowledge spillovers is reflected by the fact that the discount term $\frac{1}{1+f'\left(X_B\left(\widehat{k}\right)\right)X'_B\left(\widehat{k}\right)}$ also appears in (83): each firm in the duopoly with knowledge spillovers internalizes the fact that the R&D investment by the other firm reduces its effective interest rate paid for investing in knowledge, the direct influence of the dynamic externality.

Yet, the market supply strategies, $Q_{A,d}\left(\widehat{k}\right)$ and $Q_{A,m}\left(\widehat{k}\right)$, differ. But the difference between these two market organizations is not the typical static market externality. It is a dynamic market externality that is added. In order to see the distinction between the static market externality and the dynamic market externality, we point out the similarities and differences between the duopoly with patent protection and the duopoly with knowledge spillovers.

The similarity with the duopoly setup with patent protection is the fact that $Q_{A,d}\left(\widehat{k}\right) = Q_{A,D}\left(\widehat{k}\right)$, the market-supply *functions* are equal. However, in equilibrium, the *input* of these two market-supply functions in the next period differ. It is $Q_{A,d}\left(k + f\left(x_A\right) + f\left(x_B\right)\right)$ for the duopoly with spillovers, whereas it is $Q_{A,D}\left(k_A + f\left(x_A\right)\right)$ for the duopoly with patent protection. Firms internalize that the dynamic externality interacts with the market organization in the future period. The duopolistic market organization induces that, in the next period, there will be both a market externality and a dynamic externality, and firms internalize the *joint* effect of the two externalities into their decision-making. In other words, firms internalize the fact that *in the duopoly case with knowledge spillovers, in addition to the dynamic externality, there is a dynamic market externality.* The difference between the dynamic market externality and the static market externality is that the dynamic market externality appears when firms' R&D investment decisions have a *direct* effect on future market supply, through their direct influence on the law of motion $\widehat{k} = k + f\left(x_A\right) + f\left(x_B\right)$, which leads to $Q_{A,d}\left(k + f\left(x_A\right) + f\left(x_B\right)\right)$. In the case of duopoly with patent protection, firms' R&D decisions have an *indirect* effect on next period's market supply, as each firm accumulates its own knowledge stock, so only a static market externality affects the firms' optimizing behavior as this is reflected by their necessary optimal condition (44).

5.1 The impact of the dynamic market externality on market supply and industry growth

In order to capture the impact of the dynamic market externality on firm behavior, we compare the two market organizations that differ only with respect to this type of externality: we compare the strategies of the two monopolies with knowledge spillovers with the strategies of the duopoly with knowledge spillovers.

In two worlds, (i) with two monopolists with knowledge spillovers, and (ii) a duopoly with knowledge spillovers, where in both market organizations the stock of knowledge is exactly the same, the decision rules

$$Q_{A,m}(k) = \left(\frac{1-\frac{1}{\eta}}{\nu}\right)^{\eta} k \ ,$$

and

$$Q_{A,d}(k) + Q_{B,d}(k) = \left(\frac{1-\frac{1}{2\eta}}{\nu}\right)^{\eta} k \ ,$$

imply that consumers in the duopolistic market enjoy more supply and a lower price, compared to consumers who live in each of the two monopolistic markets. The higher aggregate supply of the duopoly is enhanced even more by the fact that $\chi_d > \chi_m$, i.e. duopolists invest more in R&D and the duopolistic industry grows more. The comparison of χ_d and χ_m appears in figure 5. Figure 5 reflects the conditions (53) and (75). Since $g_d(\chi) = g_m(\chi)$, $\kappa_m > \kappa_d \Leftrightarrow \chi_m < \chi_d$. After some algebra, it is,

$$\kappa_m > \kappa_d \Leftrightarrow h(\eta) \equiv \eta \frac{1-\left(\frac{1}{4}\right)^{\frac{1}{\eta-1}}}{1-\frac{1}{2}\left(\frac{1}{4}\right)^{\frac{1}{\eta-1}}} > 1 \ .$$

It is easy to check that $h(1) = 1$ and $h'(\eta) > 0$, for all $\eta > 1$. Therefore, it is always the case that $\kappa_m > \kappa_d$ and $\gamma_m = 2\chi_m < 2\chi_d = \gamma_d$. In brief, *the dynamic market externality increases industry growth.*

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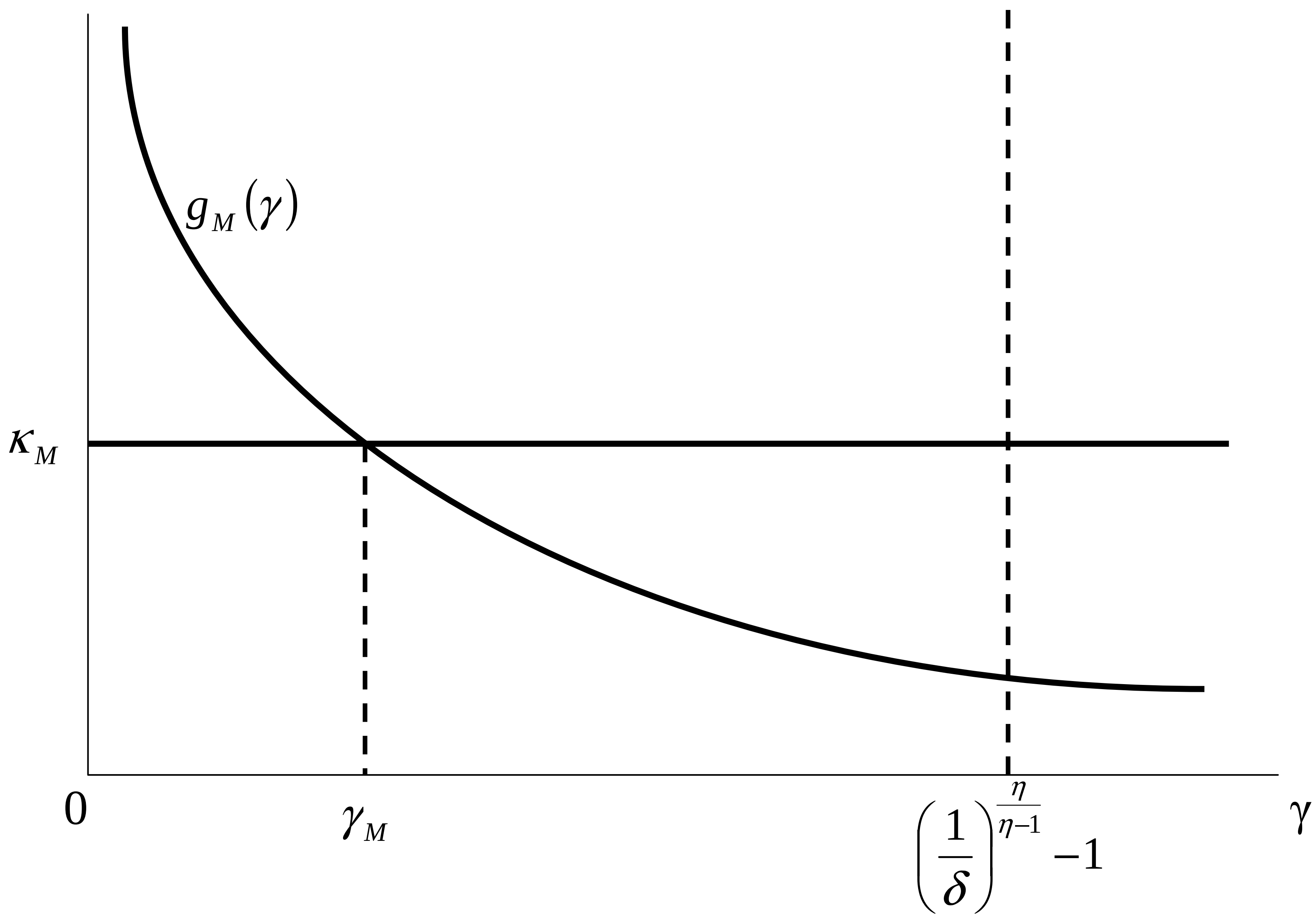

Figure 1: Benchmark Dynamic Monopoly

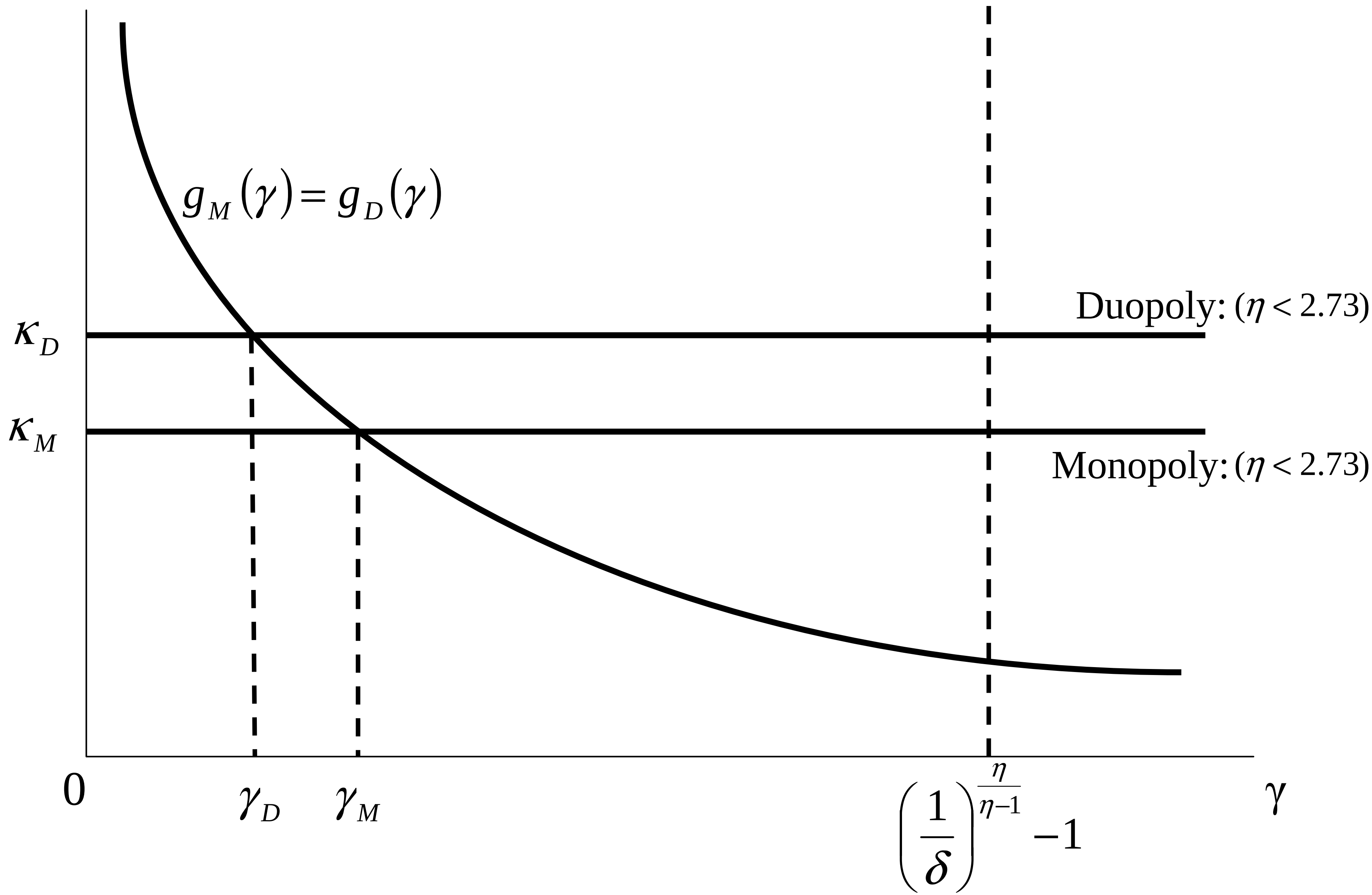

Figure 2.a: Comparison between the benchmark dynamic monopoly and duopoly with patent protection: case where $\eta < 2.73$

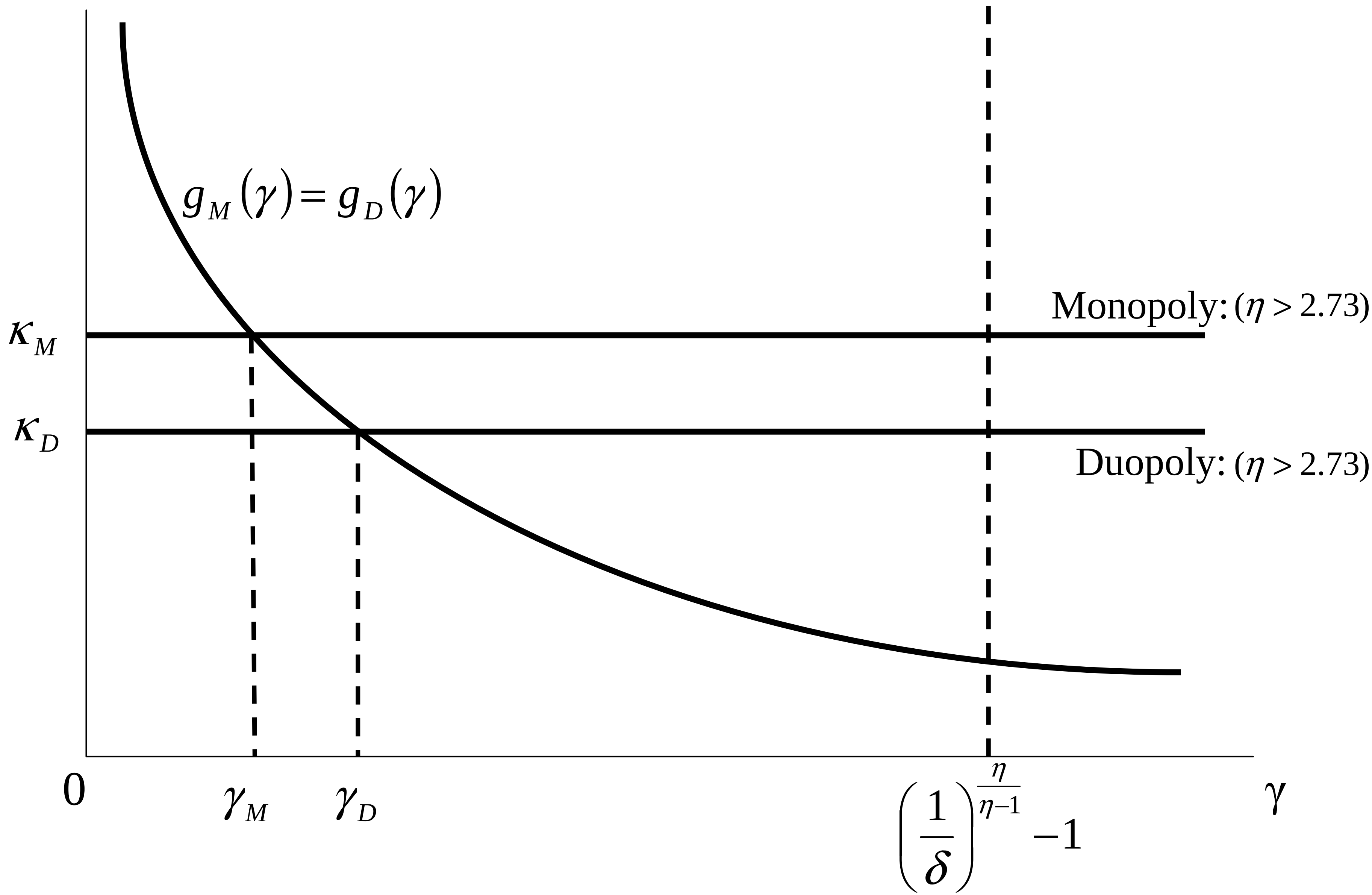

Figure 2.b: Comparison between the benchmark dynamic monopoly and duopoly with patent protection: case where $\eta > 2.73$

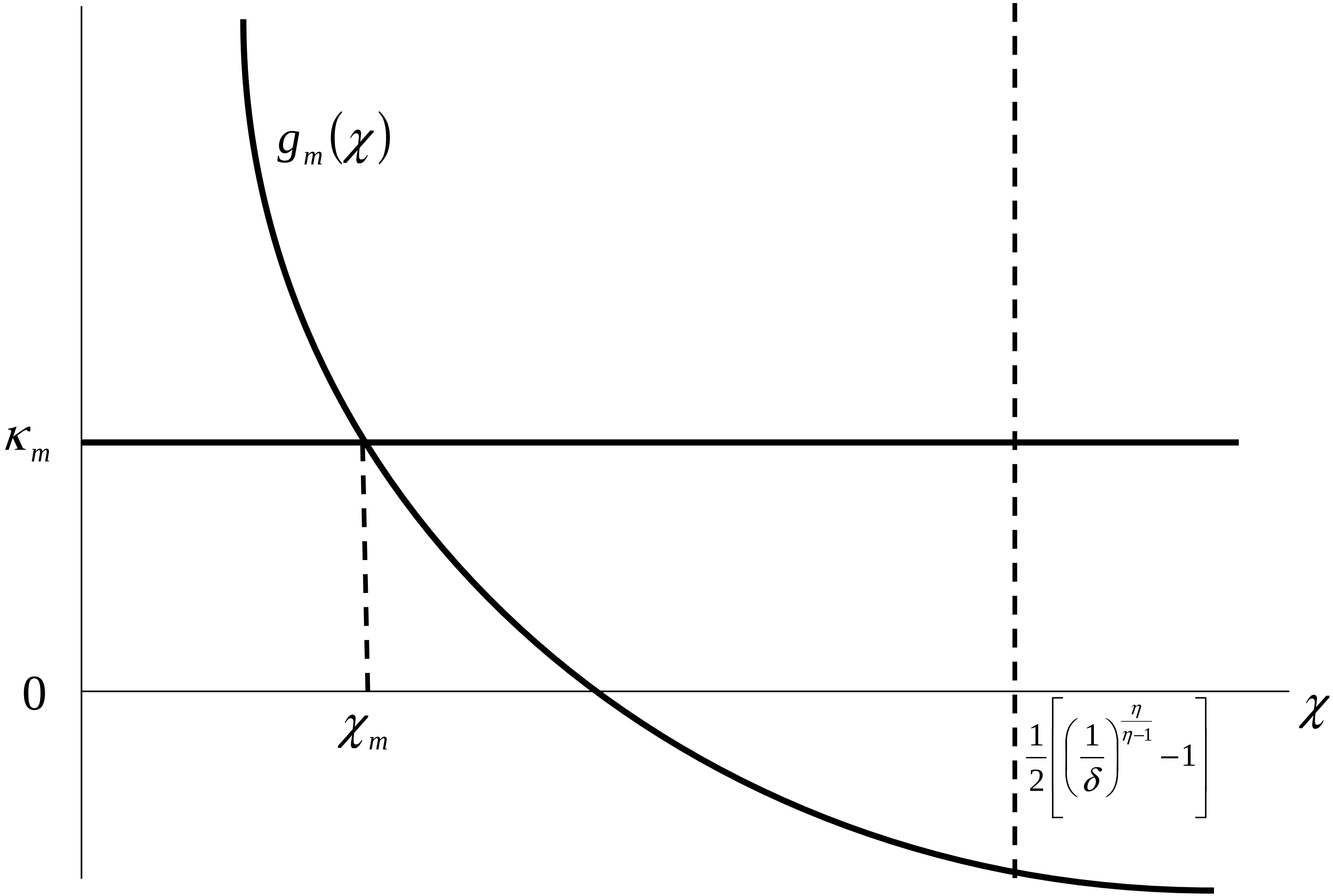

Figure 3: Dynamic Monopoly with knowledge spillovers

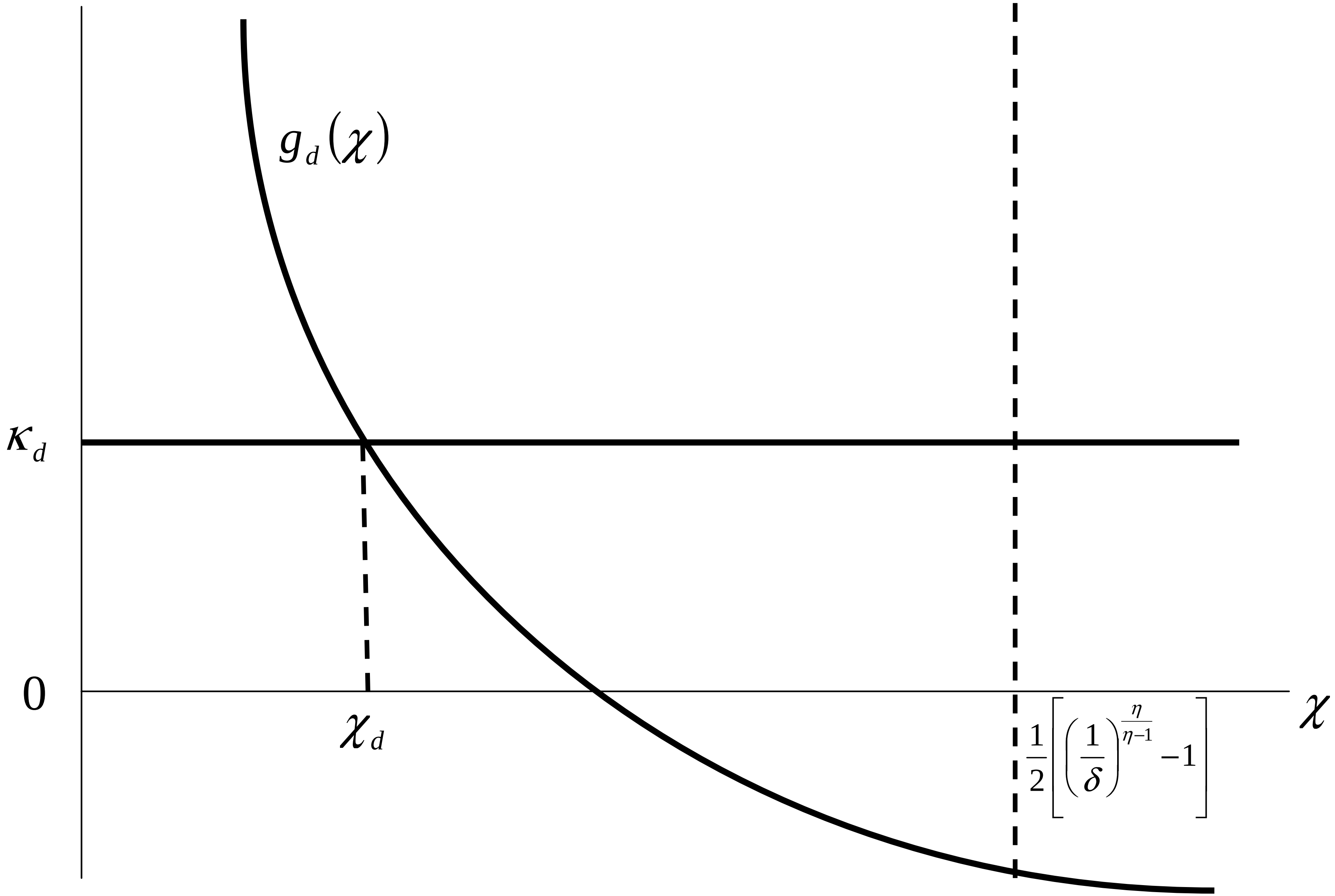

Figure 4: Dynamic duopoly with knowledge spillovers

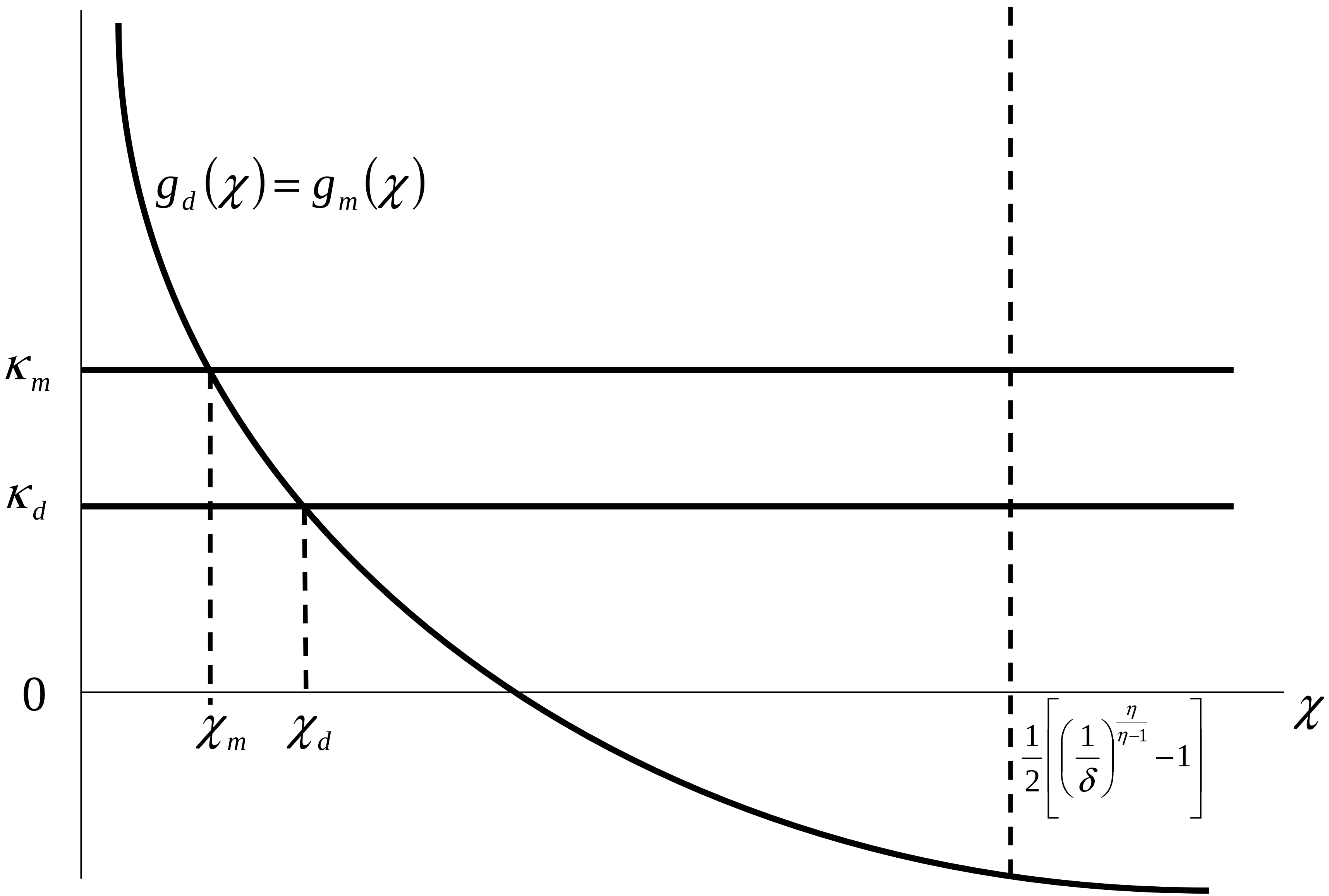

Figure 5: Comparison between the R&D strategies of the two dynamic monopolies and the duopoly with knowledge spillovers